International Journal of Theoretical Physics, Vol. 41, No. 5, May 2002 (© 2002)

Aharonov–Bohm Effect, Flat Connections, and Green's Theorem

M. A. Aguilar[1,3] and M. Socolovsky[2]

Received October 21, 2001

The validity of Green's theorem, and hence of Stokes' theorem, when the involved vector field is differentiable but not continuously differentiable, is crucial for a theoretical explanation of the Aharonov–Bohm (A-B) effect; we review this theorem. We describe the principal bundle in which the A-B effect occurs, and give the geometrical description of the relevant connection. We study the set of gauge equivalence classes of flat connections on a product bundle with abelian structural group, and show that this set has a canonical group structure, which is isomorphic to a quotient of cohomology groups. We apply this result to the A-B bundle and calculate the holonomy groups of all flat connections.

KEY WORDS: Aharonov–Bohm; flat connections; holonomy groups.

1. INTRODUCTION

The Aharonov–Bohm (A-B) effect (Aharonov and Bohm, 1959) is one of the simplest and at the same time most important examples of the interplay among classical gauge theory, quantum mechanics, differential geometry, and topology. This can be easily understood from the fact that the A-B effect is the result of the action of a nontrivial flat connection (a magnetic potential) on a section (the wave function of the electron) in a principal (product) bundle over a nonsimply connected region (ordinary Euclidean space minus a tube). Green's theorem in the plane, or its three-dimensional version, the Stokes' theorem, play a crucial role in its theoretical explanation, relating the paths followed by the charged particles in the free field region to the flux of the magnetic field in regions not accesible

[1] Instituto de Matemáticas, Universidad Nacional Autónoma de México, Cd. Universitaria, México, D.F., México.

[2] Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Cd. Universitaria, México, D.F., México.

[3] To whom correspondence should be addressed at Instituto de Matemáticas, Universidad Nacional Autónoma de México, Cd. Universitaria, 04510 México, D.F., México; e-mail: marcelo@math.unam.mx.

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to the particles. The usual conditions under which the Stokes' theorem is proved, however, are not all fulfilled by the physical system, since in the standard proof of the theorem it is assumed that the vector field, in this case the vector potential, has continuous first-order partial derivatives, while in the physical system these derivatives exist everywhere, but have a finite discontinuity at the surface of the solenoid.

There are several proofs of the generalization of Green's theorem, and therefore also of Stokes' theorem, under different assumptions. The most appropiate for our purposes is that of Craven (1964). For completeness, and for the benefit of the reader, in Section 2 we present a simplified proof of the theorem, adequate to our purposes. (In the book of Arfken and Weber, 2001, the validity of Stokes' theorem under these more general conditions is mentioned, but not explicitly proved.)

In Section 3 we review some geometrical aspects related to the A-B effect. A symmetry argument allows to neglect the longitudinal dimension along the solenoid, and then we show that the principal bundle where the effect occurs is the product $U(1)$-bundle $U(1) \to \mathbb{R}^{2*} \times U(1) \to \mathbb{R}^{2*}$, where $\mathbb{R}^{2*} = \mathbb{R}^2 - \{0\}$ is the plane minus a point (or the plane minus a disk).

Section 4 is a long section devoted to the study of the general theory of connections on a product bundle, and in particular of the subspace of flat connections in the case of a connected abelian structural group G. Our main result (Theorem 2) is that the moduli space of gauge equivalence classes of flat connections on the bundle $M \times G \to M$ is isomorphic to $H^1_{\mathrm{DR}}(M; \mathbf{g})/[M, G]$, which is the quotient of the first De Rham cohomology group of M with coefficients in the Lie algebra $\mathbf{g}$ of G, modulo the group of smooth homotopy classes of maps from M to G. In particular, this shows that the A-B effect is caused by the nontrivial topology of the base space M, because even if $G = \mathbb{R}$, then we still have flat connections which are not gauge equivalent to the canonical one. For the case $G = U(1)$ and $M = \mathbb{R}^2 - \{0\}$ we show, in Theorem 3, that the moduli space is isomorphic to S^1.

In Section 5 we calculate the holonomy group of each equivalence class of flat connections on $M \times U(1) \to M$, where $M = \mathbb{R}^2 - \{0\}$. By the isomorphism mentioned above, we can associate to the gauge class of a flat connection an element in S^1. Then the holonomy group of the connection is the subgroup of S^1 generated by this element (Theorem 4). These subgroups are either finite cyclic groups or an infinite cyclic group which is dense in S^1. This means the following: take a circle with center at the origin in $\mathbb{R}^2$; then there are potentials such that if we go around this circle n times, then its holonomy is zero. And there are potentials such that we can get as close as we want to any given value of the holonomy phase, provided we go around the circle a sufficiently large number of times.

In Section 6 we particularize the results of Section 4 to the A-B connection.

Finally, we comment about the appearance of the A-B field (6.1) in vortex solutions to the abelian Higgs model.

2. GENERALIZED GREEN'S AND STOKES' THEOREMS

In this section we prove Green's theorem in the plane under the less restrictive condition of not requiring the continuity of the first-order partial derivatives of the vector field. In so doing, we follow the proof of the Cauchy Integral Theorem by Goursat (Churchill and Brown, 1984) in the context of complex variables.

Theorem 1. *Let $\vec{A} = (A_x, A_y)$ be a differentiable but not necessarily continuously differentiable vector field in $\mathbb{R}^2$, and $\mathcal{R}$ a closed region in the plane with boundary c, a positively oriented simple piecewise smooth closed curve. If the Riemann integral of the difference of the partial derivatives $A_{y,x} - A_{x,y}$ exists on $\mathcal{R}$, then this integral equals the line integral of $\vec{A}$ along c. In particular, the integral exists if the set of points of discontinuity of the integrand has area zero.*

Proof: Let A_x and A_y be the Cartesian components of a differentiable vector field $\vec{A}$ in $\mathbb{R}^2$: by assumption all partial derivatives $\partial_x A_y = A_{y,x}$, etc., exist, but these are not necessarily continuous functions of their arguments. Let $\mathcal{R}$ be a closed region in the xy-plane with boundary $\partial\mathcal{R} = c$, a positively oriented non-self-intersecting (simple) piecewise smooth closed curve. By a lemma of Goursat (Churchill and Brown, 1984), which *mutatis mutandis* can be transferred from the context of complex analytic functions to the present case of real-valued differentiable functions, since it only involves the definition of the derivative, for any $\varepsilon > 0$ there exists a finite subdivision of $\mathcal{R}$ in closed squares and partial squares (squares with the portion outside $\mathcal{R}$ removed) $\mathcal{R}_\alpha$, $\alpha = 1, 2, \ldots, n$, with positively oriented boundaries c_α, and points $\vec{x}_\alpha \in \mathcal{R}_\alpha$ for each α, such that for all $\vec{x} \neq \vec{x}_\alpha$ in $\mathcal{R}_\alpha$, the quantities

$$\bar{\delta}_{x,y} = \frac{A_x - \bar{A}_x}{y - \bar{y}} - \bar{A}_{x,y} \tag{2.1}$$

and a similar definition for $\bar{\delta}_{y,x}$, satisfy

$$|\bar{\delta}_{x,y}|, |\bar{\delta}_{y,x}| < \varepsilon, \tag{2.2}$$

where $(\bar{x}, \bar{y}) = \vec{x}_\alpha$, $(x, y) = \vec{x}$, $A_x = A_x(x, y)$, $\bar{A}_x = A_x(\bar{x}, \bar{y})$, and $\bar{A}_{x,y} = \frac{\partial A_x}{\partial y}|_{\vec{x}_\alpha}$. For $\vec{x} = \vec{x}_\alpha$, $\bar{\delta}_{x,y} = \bar{\delta}_{y,x} = 0$, and so $\bar{\delta}_{x,y}$ and $\bar{\delta}_{y,x}$ are continuous functions of $\vec{x}$, i.e., $\lim_{\vec{x}\to\vec{x}_\alpha} \bar{\delta}_{x,y} = \lim_{\vec{x}\to\vec{x}_\alpha} \bar{\delta}_{y,x} = 0$. From (2.1),

$$A_x = \bar{A}_x + \bar{A}_{x,y}(y - \bar{y}) + \bar{\delta}_{x,y}(y - \bar{y})$$
$$A_y = \bar{A}_y + \bar{A}_{y,x}(x - \bar{x}) + \bar{\delta}_{y,x}(x - \bar{x})$$

and then

$$A_x\,dx + A_y\,dy = \bar{A}_x\,dx + \bar{A}_y\,dy + \bar{A}_{x,y}(y - \bar{y})\,dx + \bar{A}_{y,x}(x - \bar{x})\,dy$$
$$+ \bar{\delta}_{x,y}(y - \bar{y})\,dx + \bar{\delta}_{y,x}(x - \bar{x})\,dy;$$

since

$$\int_{c_\alpha} (\overline{A}_x\, dx + \overline{A}_y\, dy) = 0$$

we obtain

$$\int_{c_\alpha} (A_x\, dx + A_y\, dy) = \int_{c_\alpha} \overline{A}_{x,y}(y - \overline{y})\, dx + \int_{c_\alpha} \overline{A}_{y,x}(x - \overline{x})\, dy$$
$$+ \int_{c_\alpha} \overline{\delta}_{x,y}(y - \overline{y})\, dx + \int_{c_\alpha} \overline{\delta}_{y,x}(x - \overline{x})\, dy.$$

(When c_α coincides with the boundary c of $\mathcal{R}$, each integral over c_α is a sum of integrals over its smooth components.) Since the internal integrals over elementary contours cancel to each other, the sum over the partition for the left-hand side gives the integral around c, the boundary of $\mathcal{R}$:

$$\sum_{\alpha=1}^{n} \int_{c_\alpha} (A_x\, dx + A_y\, dy) = \oint_c (A_x\, dx + A_y\, dy).$$

Then

$$\left| \oint_c (A_x\, dx + A_y\, dy) - \sum_{\alpha=1}^{n} \left(\overline{A}_{x,y} \int_{c_\alpha} (y - \overline{y})\, dx + \overline{A}_{y,x} \int_{c_\alpha} (x - \overline{x})\, dy \right) \right|$$
$$= \left| \sum_{\alpha=1}^{n} \left(\int_{c_\alpha} \overline{\delta}_{x,y}(y - \overline{y})\, dx + \int_{c_\alpha} \overline{\delta}_{y,x}(x - \overline{x})\, dy \right) \right|$$
$$\leq \sum_{\alpha=1}^{n} \left(\left| \int_{c_\alpha} \overline{\delta}_{x,y}(y - \overline{y})\, dx \right| + \left| \int_{c_\alpha} \overline{\delta}_{y,x}(x - \overline{x})\, dy \right| \right)$$
$$\leq \sum_{\alpha=1}^{n} \left(\int_{c_\alpha} |\overline{\delta}_{x,y}(y - \overline{y})|\, dx + \int_{c_\alpha} |\overline{\delta}_{y,x}(x - \overline{x})|\, dy \right)$$
$$\leq \varepsilon \sum_{\alpha=1}^{n} \left(\int_{c_\alpha} |y - \overline{y}|\, dx + \int_{c_\alpha} |x - \overline{x}|\, dy \right); \tag{2.3}$$

since at each square or partial square, $|y - \overline{y}|, |x - \overline{x}| \leq \sqrt{2} s_n$, where s_n is the length of the sides of the nth squares, for the integrals at the extreme right of (2.3) we have $\int_{c_\alpha} |y - \overline{y}|\, dx \leq \sqrt{2} s_n \int_{c_\alpha} dx \leq 2\sqrt{2} s_n^2$, $\int_{c_\alpha} |x - \overline{x}|\, dy \leq \sqrt{2} s_n \int_{c_\alpha} dy \leq 2\sqrt{2} s_n^2$, and therefore this right-hand side is smaller than or equal to $4\sqrt{2}\varepsilon \sum_{\alpha=1}^{n} s_n^2 \leq 4\sqrt{2}\varepsilon \mathcal{A}$, where $\mathcal{A}$ is the area of a rectangle which contains $\mathcal{R}$. Since this happens for all $\varepsilon > 0$ and $\varepsilon \to 0$ as $n \to \infty$, in this limit the extreme left side

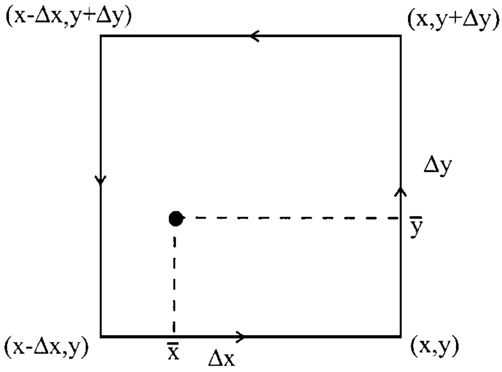

Fig. 1.

of (2.3) equals zero. On the other hand, on each square of side s_n,

$$\begin{aligned}\int_{c_\alpha}(y-\bar{y})\,dx &= \int_{\rightarrow}(y-\bar{y})\,dx + \int_{\leftarrow}(y+(\Delta y)_\alpha-\bar{y})\,dx\\ &= (y-\bar{y})(\Delta x)_\alpha + (y+(\Delta y)_\alpha-\bar{y})(-\Delta x)_\alpha = -(\Delta x)_\alpha(\Delta y)_\alpha,\\ \int_{c_\alpha}(x-\bar{x})\,dy &= \int_{\uparrow}(x-\bar{x})\,dy + \int_{\downarrow}(x-\bar{x}-(\Delta x)_\alpha)\,dy\\ &= (x-\bar{x})(\Delta y)_\alpha + (x-\bar{x}-(\Delta x)_\alpha)(-\Delta y)_\alpha = (\Delta x)_\alpha(\Delta y)_\alpha,\end{aligned}$$

where $(\Delta x)_\alpha = (\Delta y)_\alpha = s_n$ (see Fig. 1); and since in the limit as $n \to \infty$ the integrals over partial squares can be approximated, with vanishing error, by integrals over squares, we have

$$\oint_c (A_x\,dx + A_y\,dy) = \lim_{n\to\infty}\sum_{\alpha=1}^{n}(\bar{A}_{y,x} - \bar{A}_{x,y})(\Delta x)_\alpha(\Delta y)_\alpha,$$

which is the definition of the Riemann integral of $A_{y,x} - A_{x,y}$ on $\mathcal{R}$. So, assuming its existence, we obtain the Green's theorem on the plane for the functions A_x and A_y:

$$\oint_c (A_x\,dx + A_y\,dy) = \int_{\mathcal{R}}\int (A_{y,x} - A_{x,y})\,dx\,dy. \tag{2.4}$$

QED

Similar Green's formulae in the planes yz and zx can be obtained for a differentiable but *not necessarily continuously differentiable* vector field $\vec{A}$ in $\mathbb{R}^3$. From them, and through the standard procedure (Apostol, 1962; Santaló, 1961), the Stokes' theorem for the field $\vec{A}$ can be proved:

$$\oint_{\gamma} \vec{A} \cdot d\vec{l} = \int\int_{\Sigma} (\nabla \times \vec{A}) \cdot \hat{n}\, d\Sigma \tag{2.5}$$

where Σ is a simple orientable piecewise smooth surface in $\mathbb{R}^3$, with unit normal vector $\hat{n}$, which can be parametrized, at each smooth piece, by continuously differentiable functions $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$ in the uv-plane, and γ, the boundary of Σ, is a positively oriented simple piecewise smooth closed curve.

3. FIBER BUNDLE INTERPRETATION

The simplest explanation of the A-B effect considers a solenoid of radius R with a uniform and nonvanishing magnetic field $\vec{B}$ in its interior, and no magnetic field outside, placed behind a screen with a double slit. A source S emits electrons which after passing the slits interfere *quantum mechanically* at various points P of a detecting screen (see Fig. 2). The magnetic field being the rotor of the vector potential, clearly shows that this last quantity has a finite discontinuity in at least one of its first partial derivatives at the surface of the solenoid.

If $\psi_{S\to P}$ is the amplitude for an electron emitted at S to be detected at P, then $\psi_{S\to P} = \psi_1 + \psi_2$, where ψ_k, $k = 1, 2$, the amplitude for the detection when the electron follows the path γ_k in the presence of the vector potential $\vec{A}$ creating the magnetic field inside the solenoid, is given by

$$\psi_k = \psi_{k0}\, e^{\frac{iq}{\hbar c}\int_{\gamma_k} \vec{A}\cdot d\vec{l}},$$

where ψ_{k0} is the amplitude in the absence of the magnetic field, and $q = -|e|$ is the charge of the electron. The probability of detection is proportional to

$$|\psi_{S\to P}|^2 = |\psi_1 + \psi_2|^2 = |\psi_{10}|^2 + |\psi_{20}|^2 + 2Re\left(\bar{\psi}_{10}\psi_{20}\, e^{\frac{2\pi i|e|}{hc}\oint_c \vec{A}\cdot d\vec{l}}\right),$$

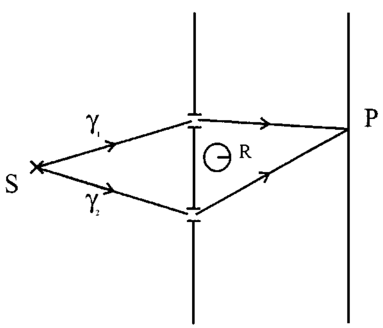

Fig. 2.

where c is the loop $\gamma_2 \cup (-\gamma_1)$; by the Stokes' theorem, the interference term is

$$2Re\left(\overline{\psi}_{10}\psi_{20}\, e^{2\pi i \frac{\Phi}{\Phi_0}}\right),$$

where Φ is the magnetic flux inside the solenoid, and $\Phi_0 = \frac{hc}{|e|}$ is the fundamental unit of magnetic flux associated with the electron. (Φ_0 is precisely the magnetic flux of the elementary Dirac monopole (Dirac, 1931) associated with the electron: for a magnetic charge $g = \frac{\hbar c}{2|e|}$, the magnetic field is $B_g(r) = \frac{\hbar c}{2|e|r^2}$, with flux through any closed surface surrounding g given by $\Phi_g = \int B_g\, dS = \frac{\hbar c}{2|e|r^2} 4\pi r^2 = \frac{hc}{|e|}$.) If $\Phi = n\Phi_0$ with $n \in \mathbb{Z}$, then there is no phase shift of the wave function of magnetic origin, i.e. no A-B effect. It is interesting to remark that at each point P of the detecting screen, the wave function is single valued; however, in general $\psi_1 \neq \psi_2$ (even $\psi_{10} \neq \psi_{20}$), and so it is the superposition principle which leads to the uniqueness of the wave function.

Clearly, the system has symmetry along the axis of the solenoid, and so the available space for the electrons, the base space M in the fiber bundle formalism, is the plane minus a disk of radius R, which is topologically equivalent to R^{2*}, the plane minus a point. The classification of principal bundles over a given space does not depend on the space itself, but only on its homotopy type, which for the present case is that of the circle, S^1. On the other hand, the structure group G of the bundle is $U(1)$, the gauge group of electromagnetism, which topologically is again the circle. So, according to the well-known theorem for the classification of G-bundles over spheres (Steenrod, 1951),

$$\begin{gathered}\left\{\begin{matrix}\text{isomorphism classes of}\\ U(1)\text{-bundles over } S^1\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{homotopy classes of}\\ \text{maps } S^0 \to S^1\end{matrix}\right\} \leftrightarrow \Pi_0(S^1)\\ = \{\text{path connected components of the circle}\} \leftrightarrow \{*\}.\end{gathered} \tag{3.1}$$

Then, up to equivalence, there is only one $U(1)$-bundle over S^1: the *product* one, that is, the torus $T^2 = S^1 \times U(1)$. (If the structure group were $O(2)$ instead of $U(1) \cong SO(2)$, then there should be an additional bundle, nontrivial and non-orientable: the Klein bottle K^2.)

The same conclusion then applies to the case $M = \mathbb{R}^{2*}$, and therefore the relevant bundle for the A-B effect is the product $U(1)$-bundle

$$\xi : U(1) \to \mathbb{R}^{2*} \times U(1) \to \mathbb{R}^{2*}. \tag{3.2}$$

For the more realistic case of Euclidean 3-space minus a tube, since $\mathbb{R}^3 - \mathbb{R}$ is also homotopically equivalent to S^1, the bundle is

$$U(1) \to (\mathbb{R}^3 - \mathbb{R}) \times U(1) \to (\mathbb{R}^3 - \mathbb{R}). \tag{3.3}$$

A useful visualization of the total space of the bundle is obtained by replacing the circle by the unit interval [0, 1] with its extreme points identified, i.e. $0 \sim 1$.

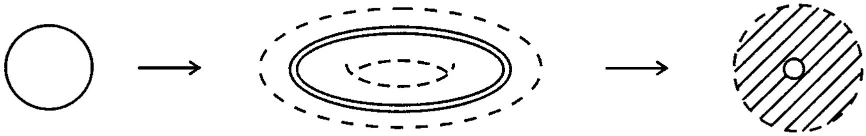

Fig. 3.

Then, the fiber bundle is

$$S^1 \to \frac{\mathbb{R}^{2*} \times [0,1]}{\sim} \to \mathbb{R}^{2*} \tag{3.4}$$

for $M = \mathbb{R}^{2*}$, and

$$S^1 \to \frac{(\mathbb{R}^3 - \mathbb{R}) \times [0,1]}{\sim} \to (\mathbb{R}^3 - \mathbb{R}) \tag{3.5}$$

in three dimensions.

Another picture of the total space of the bundle is that of an open solid 2-torus minus a circle $(T^2_\circ)^*$ since the base space $\mathbb{R}^{2*}$ is homeomorphic to an open 2-disk minus a point, $(D^2_\circ)^*$, and $(T^2_\circ)^* = (D^2_\circ)^* \times S^1$; so the bundle is

$$S^1 \to \left(T^2_\circ\right)^* \to \left(D^2_\circ\right)^* \tag{3.6}$$

(see Fig. 3), and in three dimensions,

$$S^1 \to \left(T^2_\circ\right)^* \times \mathbb{R} \to \left(D^2_\circ\right)^* \times \mathbb{R}. \tag{3.7}$$

In the following, we shall restrict ourselves to a two-dimensional base space.

4. CONNECTIONS ON THE PRODUCT BUNDLE

In this section we will study the affine space of connections on a product bundle $M \times G \to M$, and the subspace of flat connections when G is abelian.

Let G be a Lie group and $p : P \to M$ a principal G-bundle. We denote by $\Omega^k_G(P; \mathbf{g})$ the real vector space of k-forms β on P with values in $\mathbf{g}$, the Lie algebra of G, such that

1. For each $a \in P$, if there is a $v_i \in (d\rho_a)_e(\mathbf{g})$, then $\beta_a(v_1, \ldots, v_i, \ldots, v_k) = 0$, where $\rho_a : G \to P$ is given by $\rho_a(g) = a \cdot g$.
2. For each $a \in P$, $g \in G$, and $v_i \in T_a P$ $(i = 1, \ldots, k)$

$$\beta_{a\cdot g}((d\gamma_g)_a(v_1), \ldots, (d\gamma_g)_a(v_k)) = Ad(g^{-1}) \circ \beta_a(v_1, \ldots, v_k),$$

where $\gamma_g : P \to P$ is given by $\gamma_g(a) = a \cdot g$.

Let $\mathcal{C}(P)$ be the set of connections on the bundle $p : P \to M$. One defines an action $\Omega^1_G(P; \mathbf{g}) \times \mathcal{C}(P) \to \mathcal{C}(P)$ by $(\beta, \omega) \mapsto \beta \cdot \omega := \beta + \omega$. It is easy to see that this action is free and transitive; therefore each fixed connection ω_0 defines a bijection from $\Omega^1_G(P;\mathbf{g})$ to $\mathcal{C}(P)$, given by $\beta \mapsto \beta + \omega_0$, so that $\mathcal{C}(P)$ is an affine space over $\Omega^1_G(P;\mathbf{g})$.

Using the adjoint action $G \times \mathbf{g} \to \mathbf{g}$, we construct the associated vector bundle $\overline{p} : P \times_G \mathbf{g} \to M$, with fiber $\mathbf{g}$. Let us denote by $\Omega^k(M; P \times_G \mathbf{g})$ the vector space of k-forms on M with values in the vector bundle $P \times_G \mathbf{g}$. It is well known that there is an isomorphism $\Omega^k_G(P;\mathbf{g}) \xrightarrow{\cong} \Omega^k(M; P \times_G \mathbf{g})$ given by $\beta \mapsto \tilde{\beta}$, where $\tilde{\beta}_x : T_xM \times \cdots \times T_xM \to \overline{p}^{-1}(x)$ is defined by $\tilde{\beta}_x(u_1, \ldots, u_k) = [a, \beta_a(\tilde{u}_1), \ldots, \beta_a(\tilde{u}_k)]$, with a any point in $p^{-1}(x) \subset P$ and $\tilde{u}_i$ any element such that $(dp)_a(\tilde{u}_i) = u_i$ for each $i = 1, \ldots, k$.

Now we consider the case of the product bundle, i.e., $\pi : P = M \times G \to M$, where π is the projection on the first coordinate. In this case the associated bundle with fiber $\mathbf{g}$ is also a product bundle, namely $M \times \mathbf{g} \to M$. There is a canonical isomorphism

$$\begin{array}{ccc} (M \times G) \times_G \mathbf{g} & \xrightarrow{\varphi} & M \times \mathbf{g} \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

given by $\varphi[(x, g), z] = (x, Ad(g)(z))$, whose inverse is given by $\varphi^{-1}(x, z) = [(x, e), z]$. Clearly, forms on M with values in $\mathbf{g}$ is the same as forms on M with values on the trivial bundle $M \times \mathbf{g}$, i.e., $\Omega^1(M;\mathbf{g}) \cong \Omega^1(M; M \times \mathbf{g})$. Thus we have the following:

Lemma 1. *There is an isomorphism* $\Omega^k(M;\mathbf{g}) \to \Omega^k_G(M \times G;\mathbf{g})$ *given by* $\gamma \mapsto \hat{\gamma}$, *where* $\hat{\gamma}_{(x,g)} : (T_xM \times T_gG) \times \cdots \times (T_xM \times T_gG) \to \mathbf{g}$ *is defined by*

$$\hat{\gamma}_{(x,g)}((u_1, v_1), \ldots, (u_k, v_k)) = Ad(g^{-1}) \circ \gamma_x(u_1, \ldots, u_k).$$

And its inverse $\Omega^k_G(M \times G;\mathbf{g}) \to \Omega^k(M;\mathbf{g})$ *is given by* $\beta \mapsto \overline{\beta}$, *where* $\overline{\beta}_x : T_xM \times \cdots \times T_xM \to \mathbf{g}$ *is defined by*

$$\overline{\beta}_x(u_1, \ldots, u_k) = \beta_{(x,e)}((u_1, 0), \ldots, (u_k, 0)).$$

Proof: Putting together the isomorphisms defined above, we get $\Omega^k(M;\mathbf{g}) \cong \Omega^k(M; M \times \mathbf{g}) \cong \Omega^k(M; (M \times G) \times_G \mathbf{g}) \cong \Omega^k_G(M \times G;\mathbf{g})$. It is easy to see that their composite sends γ to $\hat{\gamma}$, and that its inverse maps β to $\overline{\beta}$. QED

Definition 1. Let G be a Lie group. There is a canonical left invariant 1-form $\mathcal{M} \in \Omega^1(G;\mathbf{g})$ defined as follows. Since it is left invariant, it is determined by its value at $e \in G$, i.e., by $\mathcal{M}_e : T_eG = \mathbf{g} \to \mathbf{g}$, where it is defined as the identity. Therefore at any other point $g \in G$, it is given by $\mathcal{M}_g(v) = (dL_{g^{-1}})_g(v)$, where $L_{g^{-1}} : G \to G$ is given by $L_{g^{-1}}(h) = g^{-1}h$.

Definition 2. Consider the product bundle $\pi : M \times G \to M$. Using the projection on the second factor $q : M \times G \to G$ and the canonical form $\mathcal{M} \in \Omega^1(G;\mathbf{g})$, we can define a canonical 1-form ω_0 on $M \times G$ with values in $\mathbf{g}$ by $\omega_0 := q^{\#}(\mathcal{M})$. It is easy to see that this is a connection on the product bundle, and by definition, $\omega_{0(x,g)}(u, v) = (dL_{g^{-1}})_g(v)$.

Lemma 2. *Let $\pi : M \times G \to M$ be the product bundle. Then there is a canonical bijection $\Omega^1(M; \mathbf{g}) \to \mathcal{C}(M \times G)$ given by $\gamma \mapsto \omega^\gamma$, where $\omega^\gamma_{(x,g)}(u, v) = Ad(g^{-1}) \circ \gamma_x(u) + (dL_{g^{-1}})_g(v)$.*

Proof: By Lemma 1, we have an isomorphism $\Omega^1(M; \mathbf{g}) \xrightarrow{\cong} \Omega^1_G(M \times G; \mathbf{g})$, $\gamma \mapsto \hat{\gamma}$. Using the canonical connection ω_0 of Definition 2, we have a bijection $\Omega^1_G(M \times G; \mathbf{g}) \to \mathcal{C}(M \times G)$ given by $\beta \mapsto \beta + \omega_0$. The composite of these gives the desired bijection $\gamma \mapsto \omega^\gamma$. QED

Lemma 3. *Let $\mathcal{G}(M \times G)$ be the gauge group of the product bundle, and let $C^\infty(M; G)$ be the group (under pointwise multiplication) of smooth maps from M to G. Then $C^\infty(M; G)$ is isomorphic to $\mathcal{G}(M \times G)$.*

Proof: We define a homomorphism $\Phi : C^\infty(M; G) \to \mathcal{G}(M \times G)$ by $\Phi(f) = \phi_f$, where $\phi_f : M \times G \to M \times G$ is given by $\phi(f)(x, g) = (x, f(x)g)$. We will show that Φ is an isomorphism. Assume that $\phi_f(x, g) = (x, g)$, for all pairs (x, g), then $f(x)g = g$ and hence $f(x) = e$, for all $x \in M$. Therefore f is the neutral element of $C^\infty(M; G)$. To see that Φ is surjective, let ϕ be a gauge transformation. Define f to be the composite $M \xrightarrow{\iota} M \times G \xrightarrow{\phi} M \times G \xrightarrow{q} G$, where $\iota(x) = (x, e)$ and $q(x, g) = g$. Since both ϕ and $\phi_f = \Phi(f)$ are G-equivariant, we have that $\phi(x, g) = \phi(x, e) \cdot g$ and $\phi_f(x, g) = \phi_f(x, e) \cdot g$. Hence, to show that they are equal, one only has to check that they coincide on elements of the form (x, e), but $\phi_f(x, e) = (x, f(x)) = \phi(x, e)$. Therefore $\Phi(f) = \phi$. QED

Given any principal G-bundle $p : P \to M$, there is an action $\mathcal{C}(P) \times \mathcal{G}(P) \to \mathcal{C}(P)$ given by $(\omega, \phi) \mapsto \omega \cdot \phi := \phi^{\#}(\omega)$.

Lemma 4. *Let $\pi : M \times G \to M$ be a product bundle, where G is an abelian Lie group. Then we have a commutative diagram*

$$\begin{array}{ccc}
\mathcal{C}(M \times G) \times \mathcal{G}(M \times G) & \longrightarrow & \mathcal{C}(M \times G) \\
\cong\uparrow & & \uparrow\cong \\
\Omega^1(M; \mathbf{g}) \times C^\infty(M; G) & \longrightarrow & \Omega^1(M; \mathbf{g}),
\end{array}$$

where $(\gamma, f) \mapsto \gamma \cdot f := \gamma + f^{\#}(\mathcal{M})$, for all $\gamma \in \Omega^1(M; \mathbf{g})$ and $f \in C^\infty(M; G)$.

Proof: The vertical isomorphisms are given by Lemmas 2 and 3; therefore we have to show that $\phi^{\#}_f(\omega^\gamma)$ is equal to $\omega^{\gamma + f^{\#}(\mathcal{M})}$. Since $\omega^\gamma = \hat{\gamma} + \omega_0$, then

$\phi_f^\#(\omega^\gamma)_{(x,g)}(u,v) = \gamma_x(u) + d(L_{f(x)^{-1}} \circ f)_x(u) + (dL_{g^{-1}})_g(v)$. On the other hand, $\omega_{(x,g)}^{\gamma+f^\#(\mathcal{M})}(u,v) = \gamma_x(u) + f^\#(\mathcal{M})_x(u) + (dL_{g^{-1}})_g(v)$. But

$$f^\#(\mathcal{M})_x(u) = \mathcal{M}_{f(x)}((df)_x(u)) = \left(dL_{f(x)^{-1}}\right)_{f(x)}((df)_x(u)).$$

Hence $\phi_f^\#(\omega^\gamma) = \omega^{\gamma+f^\#(\mathcal{M})}$. QED

We will now consider the curvature of the connections on the product bundle.

Lemma 5. *The connection ω_0 on the bundle $\pi : M \times G \to M$ is flat.*

Proof: It is well known that a connection is flat if and only if its distribution of horizontal spaces is integrable. By Definition 2, $\omega_{0(x_0,g_0)} : T_{x_0}M \times T_{g_0}G \to \mathbf{g}$ is given by $\omega_{0(x_0,g_0)}(u,v) = (dL_{g_0^{-1}})_{g_0}(v)$; therefore, $\ker(\omega_{0(x_0,g_0)}) = \{(u,0) \in T_{x_0}M \times T_{g_0}G\}$. Now for each $g_0 \in G$, consider the embedding $\iota_{g_0} : M \to M \times G$ given by $\iota_{g_0}(x) = (x, g_0)$. Then $(dL_{g_0})_{x_0} : T_{x_0}M \to T_{x_0}M \times T_{g_0}G$ and clearly the image of $(dL_{g_0})_{x_0}$ is the subspace $\{(u,0) \in T_{x_0}M \times T_{g_0}G\}$. Therefore the horizontal distribution is integrable. QED

Proposition 1. *Let G be an abelian Lie group and consider the bundle $\pi : M \times G \to M$. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{C}(M \times G) & \xrightarrow{curvature} & \Omega_G^2(M \times G; \mathbf{g}) \\ \cong\uparrow & & \downarrow\cong \\ \Omega^1(M; \mathbf{g}) & \xrightarrow{d} & \Omega^2(M; \mathbf{g}), \end{array}$$

where the function at the top maps a connection ω to its curvature F^ω and d is the exterior derivative.

Proof: The isomorphism on the left is given by Lemma 2, and so if $\gamma \in \Omega^1(M; \mathbf{g})$, then $\omega^\gamma = \hat{\gamma} + \omega_0$, and, since G is abelian, $\hat{\gamma}_{(x,g)}(u,v) = \gamma_x(u)$. Furthermore, by the first structural equation, $F^{\omega_0} = d\omega_0 + 1/2[\omega_0, \omega_0]$, so that $F^{\omega_0} = d\omega_0$. By Lemma 5, $F^{\omega_0} = 0$; hence, $d\omega_0 = 0$.

Now let us consider the image of F^{ω^γ} under the isomorphism on the right-hand side which is given by Lemma 1, namely $F^{\omega^\gamma} \mapsto \overline{F}^{\omega^\gamma}$, where $\overline{F}_x^{\omega^\gamma}(u_1,u_2) = F_{(x,e)}^{\omega^\gamma}((u_1,0),(u_2,0))$ which is equal, by the first structural equation, to

$$d\omega_{(x,e)}^\gamma((u_1,0),(u_2,0)).$$

Since $\omega^\gamma = \hat{\gamma} + \omega_0$, and $d\omega_0 = 0$, then

$$d\omega_{(x,e)}^\gamma((u_1,0),(u_2,0)) = d\hat{\gamma}_{(x,e)}((u_1,0),(u_2,0)).$$

But $\hat{\gamma}(x, g)(u, v) = \gamma_x(u)$, i.e. $\hat{\gamma} = \pi^{\#}(\gamma)$; therefore

$$\begin{aligned} d\hat{\gamma}_{(x,e)}((u_1, 0), (u_2, 0)) &= d\pi^{\#}(\gamma)_{(x,e)}((u_1, 0), (u_2, 0)) \\ &= (\pi^{\#} d\gamma)_{(x,e)}((u_1, 0), (u_2, 0)) \\ &= (d\gamma)_x\big((d\pi)_{(x,e)}(u_1, 0), (d\pi)_{(x,e)}(u_2, 0)\big) \\ &= (d\gamma)_x(u_1, u_2). \end{aligned}$$

Hence the diagram commutes. QED

Corollary 1. *Let G be an abelian Lie group and $\pi : M \times G \to M$ a product bundle. Then there is an isomorphism between the 1-cocycles on M with coefficients in $\mathbf{g}$ and the vector space of flat connections on π, given by $\gamma \mapsto \omega^{\gamma}$.*

Proof: We give $\mathcal{C}(M \times G)$ a vector space structure using the canonical bijection of Lemma 2; in this structure the neutral element is ω_0. Since the space of 1-cocycles on M with coefficients in $\mathbf{g}$ is the kernel of d, the result is immediate from the commutativity of the diagram of Proposition 1. QED

Proposition 2. *Let G be an abelian Lie group and $\pi : M \times G \to M$ a product bundle. Then all flat connections ω on π are of the form $\omega = \omega^{d_0 f}$, where $f : M \to \mathbf{g}$ is a smooth map, if and only if every homomorphism from $\Pi_1(M)$ to $\mathbb{R}$ is zero.*

Proof: Consider the De Rham complex with coefficients in $\mathbf{g}$:

$$\Omega^0(M; \mathbf{g}) \xrightarrow{d_0} \Omega^1(M; \mathbf{g}) \xrightarrow{d_1} \Omega^2(M; \mathbf{g}) \to \cdots.$$

Since $H^1_{\mathrm{DR}}(M; \mathbf{g}) = \ker(d_1)/\mathrm{Im}(d_0)$, then $H^1(M; \mathbf{g}) = 0$ if and only if

$$\ker(d_1) = \mathrm{Im}(d_0);$$

and by Corollary 1, the map from $\ker(d_1)$ to the subspace of flat connections given by $\gamma \mapsto \omega^{\gamma}$ is an isomorphism. But $\mathbf{g} \cong \mathbb{R}^{\mathbf{m}}$, for some m; hence $H^1_{\mathrm{DR}}(M; \mathbf{g}) \cong H^1_{\mathrm{DR}}(M; \mathbb{R}) \oplus \cdots \oplus H^1_{\mathrm{DR}}(M; \mathbb{R})$. Therefore $H^1_{\mathrm{DR}}(M; \mathbf{g}) = 0$ if and only if

$$H^1_{\mathrm{DR}}(M; \mathbb{R}) = 0.$$

By De Rahm's theorem $H^1_{\mathrm{DR}}(M; \mathbb{R}) \cong H^1(M; \mathbb{R})$, and by the Universal Coefficient Theorem $H^1(M; \mathbb{R}) \cong Hom_{\mathbb{Z}}(H_1(M; \mathbb{Z}), \mathbb{R})$. Since $\mathbb{R}$ is abelian and $H_1(M; \mathbb{Z})$ is the abelianization of $\Pi_1(M)$, we have that

$$Hom_{\mathbb{Z}}(H_1(M; \mathbb{Z}), \mathbb{R}) \cong Hom_{\mathbb{Z}}(\Pi_1(M); \mathbb{R}).$$ QED

Now we will study the gauge equivalence classes of flat connections on a product bundle with abelian structural group.

Lemma 6. *Let G be a connected abelian Lie group and $\mathcal{M} \in \Omega^1(G; \mathbf{g})$ its canonical 1-form (see Definition 1). Then $d_1\mathcal{M} = 0$.*

Proof: Let X, Y be in $\mathbf{g}$; since $\mathcal{M}$ is left invariant, then

$$d_1\mathcal{M}(X, Y) = -\mathcal{M}[X, Y].$$

By Warner (1983) $\mathbf{g}$ is abelian; hence $d_1\mathcal{M}(X, Y) = 0$. Now let v_1, v_2 be in $T_g G$, then $v_i = (dL_g)_e(Y_i)$, $i = 1, 2$, where $Y_i \in T_e G$. Let $\tilde{Y}_i$ be the left invariant vector field generated by Y_i, so that $(\tilde{Y}_i)_g = v_i$. Then $(d_1\mathcal{M})_g(v_1, v_2) = (d_1\mathcal{M})_g((\tilde{Y}_1)_g, (\tilde{Y}_2)_g) = d_1\mathcal{M}(\tilde{Y}_1, \tilde{Y}_2)(g) = 0$. QED

Corollary 2. *Let G be a connected abelian Lie group and $\pi : M \times G \to M$ a product bundle. Then the action $\Omega^1(M; \mathbf{g}) \times C^\infty(M; G) \to \Omega^1(M; \mathbf{g})$ given by $\gamma \cdot f = \gamma + f^\#(\mathcal{M})$, leaves $\mathcal{Z}^1(M; \mathbf{g}) \subset \Omega^1(M; \mathbf{g})$ invariant.*

Proof: Let γ be in $\Omega^1(M; \mathbf{g})$ and f in $C^\infty(M; \mathbf{g})$. Then $d_1(\gamma \cdot f) = d_1(\gamma + f^\#(\mathcal{M})) = d_1(\gamma) + f^\#(d_1\mathcal{M})$. By Lemma 6, $d_1\mathcal{M} = 0$; therefore $d_1(\gamma \cdot f) = d_1\gamma$. Hence, γ is a cocycle if and only if $\gamma \cdot f$ is a cocycle. QED

Lemma 7. *Let G be a connected abelian Lie group and $\pi : M \times G \to M$ a product bundle. Then the set of equivalence classes $\mathcal{Z}^1(M; \mathbf{g})/C^\infty(M; \mathbf{g})$ has a canonical group structure.*

Proof: By Corollary 2, we have an action

$$\mathcal{Z}^1(M; \mathbf{g}) \times C^\infty(M; G) \to \mathcal{Z}^1(M; \mathbf{g})$$

given by $\gamma \cdot f = \gamma + f^\#(\mathcal{M})$. Define $J : C^\infty(M; G) \to \mathcal{Z}^1(M; \mathbf{g})$ by $J(f) = f^\#(\mathcal{M})$. By Lemma 4, the map $(\gamma, f) \mapsto \gamma \cdot f = \gamma + f^\#(\mathcal{M})$ is an action, thus if f_1, f_2 are in $C^\infty(M; G)$ then $\gamma \cdot (f_1 f_2) = (\gamma \cdot f_1) \cdot f_2$, i.e. $\gamma + (f_1 f_2)^\#(\mathcal{M}) = \gamma + f_1^\#(\mathcal{M}) + f_2^\#(\mathcal{M})$; therefore $(f_1 f_2)^\#(\mathcal{M}) = f_1^\#(\mathcal{M}) + f_2^\#(\mathcal{M})$, which means that J is a homomorphism. Since $\mathcal{Z}^1(M; \mathbf{g})/C^\infty(M; G) = \mathcal{Z}^1(M; \mathbf{g})/J(C^\infty(M; G))$ and the right-hand side is a quotient group, we obtain the canonical group structure. QED

Definition 3. Let M be a smooth manifold and G a Lie group. We denote by $[M, G]$ the set of smooth homotopy classes of smooth maps from M to G. Given a smooth map $f : M \to G$, we denote by $[f]$ its smooth homotopy class. It is easy to check that there is a group structure on $[M, G]$, given by $[f_1] \cdot [f_2] = [f_1 f_2]$, where $(f_1 f_2)(x) = f_1(x) f_2(x)$.

Now assume that G is a connected abelian Lie group. By Lemma 6, $d_1\mathcal{M} = 0$, and so we can take $[\mathcal{M}] \in H^1_{\mathrm{DR}}(G; \mathbf{g})$.

We define $\iota : [M, G] \to H^1_{\mathrm{DR}}(M; \mathbf{g})$ by $\iota[f] = f^*[\mathcal{M}] = [f^\#(\mathcal{M})]$; this is well defined since two smoothly homotopic maps induce the same homomorphism in De Rham cohomology (Warner, 1983). By the proof of Lemma 7, $(f_1 f_2)^\#(\mathcal{M}) = f_1^\#(\mathcal{M}) + f_2^\#(\mathcal{M})$; therefore $\iota([f_1] \cdot [f_2]) = \iota[f_1 f_2] = (f_1 f_2)^*[\mathcal{M}] = [(f_1 f_2)^\# (\mathcal{M})] = [f_1^\#(\mathcal{M}) + f_2^\#(\mathcal{M})] = [f_1^\#(\mathcal{M})] + [f_2^\#(\mathcal{M})] = f_1^*[\mathcal{M}] + f_2^*[\mathcal{M}] = \iota[f_1] + \iota[f_2]$. Hence ι is a homomorphism.

Definition 4. Let G be an abelian Lie group and $\pi : M \times G \to M$ a product bundle. We will define a canonical group structure on the set $\mathcal{C}(M \times G)/\mathcal{G}(M \times G)$ of gauge equivalence classes of connections on π. In order to do it, recall that, by Lemma 2, we have a bijection $\Omega^1(M; \mathbf{g}) \to \mathcal{C}(M \times G)$ given by $\gamma \mapsto \hat{\gamma} + \omega_0 = \pi^\#(\gamma) + \omega_0$, which can be used to define a group structure on $\mathcal{C}(M \times G)$ by setting $(\hat{\gamma}_1 + \omega_0) * (\hat{\gamma}_2 + \omega_0) := (\hat{\gamma}_1 + \hat{\gamma}_2) + \omega_0$, with ω_0 as the neutral element. By Lemma 3, each gauge transformation is of the form ϕ_f for a unique map $f : M \to G$. Using Lemma 4 it follows that $\phi_f^*(\omega_0) = f^\#(\mathcal{M}) + \omega_0 = \pi^\#(f^\#(\mathcal{M})) + \omega_0 = (f \circ \pi)^\#(\mathcal{M}) + \omega_0 (*)$. Now we define $j : \mathcal{G}(M \times G) \to \mathcal{C}(M \times G)$ by $j(\phi_f) := \phi_f^\#(\omega_0)$. By Lemma 3, $\phi_{f_1} \circ \phi_{f_2} = \phi_{f_1 f_2}$; by Lemma 4, $(f_1 f_2)^\#(\mathcal{M}) = f_1^\#(\mathcal{M}) + f_2^\#(\mathcal{M})$; then, using $(*)$, $j(\phi_{f_1} \circ \phi_{f_2}) = j(\phi_{f_1 f_2}) = \phi^\#_{f_1 f_2}(\omega_0) = (f_1 f_2 \circ \pi)^\# (\mathcal{M}) + \omega_0 = \pi^\#((f_1 f_2)^\#(\mathcal{M})) + \omega_0 = \pi^\# f_1^\#(\mathcal{M}) + \pi^\# f_2^\#(\mathcal{M}) + \omega_0 = [(f_1 \circ \pi)^\# (\mathcal{M}) + \omega_0] * [(f_2 \circ \pi)^\#(\mathcal{M}) + \omega_0] = j(\phi_{f_1}) * j(\phi_{f_2})$. Hence j is a homomorphism and we can form the quotient group $\mathcal{C}(M \times G)/j(\mathcal{G}(M \times G))$. By definition, the equivalence relation to form this group is the following: $\hat{\gamma}_1 + \omega_0 \sim \hat{\gamma}_2 + \omega_0 \Leftrightarrow$ there exists a smooth map $f : M \to G$ such that $(\hat{\gamma}_1 - \hat{\gamma}_2) + \omega_0 = j(\phi_f) := \phi_f^\#(\omega_0)$.

On the other hand, the equivalence relation to form $\mathcal{C}(M \times G)/\mathcal{G}(M \times G)$ is given by $\hat{\gamma}_1 + \omega_0 \sim \hat{\gamma}_2 + \omega_0 \Leftrightarrow$ there exists a smooth map $f : M \to G$ such that $\hat{\gamma}_1 + \omega_0 = \phi_f^\#(\hat{\gamma}_2 + \omega_0)$. Now, $\phi_f^\#(\hat{\gamma}_2 + \omega_0) = \phi_f^\#(\hat{\gamma}_2) + \phi_f^\#(\omega_0)$ and $\phi_f^\#(\hat{\gamma}_2) = \phi_f^\#(\pi^\#(\gamma_2)) = (\pi \circ \phi_f)^\#(\gamma_2)$, but $\pi \circ \phi_f = \pi$, hence $\phi_f^\#(\hat{\gamma}_2) = \pi^\#(\gamma_2) = \hat{\gamma}_2$. Therefore, this relation is $\hat{\gamma}_1 + \omega_0 = \hat{\gamma}_2 + \phi_f^\#(\omega_0)$. So both equivalence relations are the same and then we have a canonical group structure.

Theorem 2. *Let G be a connected abelian Lie group and $\pi : M \times G \to M$ a product bundle. Let Flat$(M \times G)$ be the vector space of flat connections on π. Then there is a canonical isomorphism (of groups)*

$$H^1_{\mathrm{DR}}(M; \mathbf{g})/\iota[M, G] \xrightarrow{\cong} \mathit{Flat}(M \times G)/\mathcal{G}(M \times G)$$

given by $\overline{[\gamma]} \mapsto \langle \omega^\gamma \rangle$, where $[\ ]$ denotes the cohomology class of γ, the bar denotes the equivalence under the action of $[M, G]$, and $\langle\ \rangle$ the gauge class of a connection.

Proof: By Corollary 1, there is an isomorphism $\mathcal{Z}^1(M; \mathbf{g}) \xrightarrow{\cong} \mathit{Flat}(M \times G)$ given by $\gamma \mapsto \omega^\gamma$. By Lemma 4, the action of the gauge group corresponds to the action of $C^\infty(M;G)$ on $\mathcal{Z}^1(M;\mathbf{g})$. Then, with the group structure of Definition 4, one has an isomorphism $\mathcal{Z}^1(M;\mathbf{g})/C^\infty(M;G) \xrightarrow{\cong} \mathit{Flat}(M\times G)/\mathcal{G}(M\times G)$, where, by Lemma 7, $\mathcal{Z}^1(M;G)/C^\infty(M;G) = \mathcal{Z}^1(M;\mathbf{g})/J(C^\infty(M;G))$, which is a quotient group.

Let $q : \mathcal{Z}^1(M;\mathbf{g}) \to H^1_{\mathrm{DR}}(M;\mathbf{g})$ be the quotient homomorphism, i.e., $q(\gamma) = [\gamma]$, and let f be in $C^\infty(M;G)$. Then $qJ(f) = [f^\#(\mathcal{M})] = f^*[\mathcal{M}] = \iota[f]$, i.e., $qJ(C^\infty(M;G)) \subset \iota[M,G]$. Therefore q induces a homomorphism

$$\bar{q} : \mathcal{Z}^1(M;\mathbf{g})/J(C^\infty(M;G)) \to H^1_{\mathrm{DR}}(M;\mathbf{g})/\iota[M,G]$$

given by $\bar{q}\sigma(\gamma) = \tau q(\gamma)$, where $\sigma : \mathcal{Z}^1(M;\mathbf{g}) \to \mathcal{Z}^1(M;\mathbf{g})/J(C^\infty(M;G))$ and $\tau : H^1_{\mathrm{DR}}(M;\mathbf{g}) \to H^1_{\mathrm{DR}}(M;\mathbf{g})/\iota[M,G]$ are the quotient homomorphisms.

Since q and τ are surjective, $\bar{q}\circ\sigma$ is surjective and hence $\bar{q}$ is surjective. Now we will show that $\bar{q}$ is injective. Clearly, $\bar{q}$ is injective if and only if $q^{-1}(\iota[M,G]) \subset J(C^\infty(M;G))$, and so we will prove this last statement. Let γ be in $q^{-1}(\iota[M,G])$, then there exists a smooth map $f : M \to G$ such that $[\gamma] = f^*(\mathcal{M}) = [f^\#]$, i.e., γ and $f^\#(\mathcal{M})$ represent the same cohomology class in $H^1_{\mathrm{DR}}(M;\mathbf{g})$, therefore there is a smooth map $\varphi : M \to \mathbf{g}$ such that $\gamma - f^\#(\mathcal{M}) = d_0(\varphi)$. Consider the exponential map $\mathit{exp} : \mathbf{g} \to G$ and define a smooth map $h : M \to G$ by $h := (\mathit{exp}\circ\varphi)f$. In order to evaluate $h^\#(\mathcal{M})$, we need to calculate the differential of exp at any point $X \in \mathbf{g}$. So take the line α in $\mathbf{g}$ defined by $\alpha(t) = X + tY$; then, using the fact that $\mathbf{g}$ is abelian, we have that $\mathit{exp}(\alpha(t)) = \mathit{exp}(X+tY) = \mathit{exp}(X)\mathit{exp}_Y(t)$. Hence $(\mathit{exp}\circ\alpha)(t) = L_{\mathit{exp}(X)}\circ\mathit{exp}_Y(t)$, where $\mathit{exp}_Y : \mathbb{R} \to G$ is the unique homomorphism such that $\dot{\mathit{exp}}_Y(0) = Y$. Therefore $(d\,\mathit{exp})_X(Y) = (\mathit{exp}\,\dot{\circ}\,\alpha)(0) = (dL_{\mathit{exp}(X)})_e(Y)$, and then

$$\begin{aligned} \mathit{exp}^\#(\mathcal{M})_X(Y) &= \mathcal{M}_{\mathit{exp}(X)}((d\,\mathit{exp})_X(Y)) \\ &= \left(dL_{\mathit{exp}(X)^{-1}}\right)_{\mathit{exp}(X)}\left(\left(dL_{\mathit{exp}(X)}\right)_e(Y)\right) = Y. \end{aligned} \tag{$*$}$$

Now, since $(f_1 f_2)^\#(\mathcal{M})\, f_1^\#(\mathcal{M}) + f_2^\#(\mathcal{M})$ (proof of Lemma 7), and using (*), we have that $h^\#(\mathcal{M})_x(v) = ((\mathit{exp}\circ\varphi)f)^\#(\mathcal{M})_x(v) = \varphi^\#(\mathit{exp}^\#(\mathcal{M}))_x(v) + f_x^\#(v) = \mathit{exp}^\#(\mathcal{M})_{\varphi(x)}((d_0\varphi)_x(v)) + f^\#(\mathcal{M})_x(v) = (d_0\varphi)_x(v) + f^\#(\mathcal{M})_x(v)$, i.e., $h^\#(\mathcal{M}) = d_0\varphi + f^\#(\mathcal{M})$. But $\gamma = d_0\varphi + f^\#(\mathcal{M})$, and hence $\gamma = h^\#(\mathcal{M})$. So, we have found a smooth map $h : M \to G$ such that $J(h) = h^\#(\mathcal{M}) = \gamma$; therefore $\gamma \in J(C^\infty(M;G))$ and then $q^{-1}(\iota[M,G]) \subset J(C^\infty(M;G))$, and $\bar{q}$ is injective. Finally, the composition of $\bar{q}$ with the isomorphism given above, maps $[\gamma]$ to $\langle\omega^\gamma\rangle$. QED

Corollary 3. *Let $\pi : M\times U(1) \to M$ be a product bundle. Then there is an isomorphism $H^1_{\mathrm{DR}}(M;\mathbb{R})/lH^1(M;\mathbb{Z}) \xrightarrow{\cong} \mathit{Flat}(M\times U(1))/\mathcal{G}(M\times U(1))$.*

Proof: Since the Lie algebra of $U(1)$ is $i\mathbb{R} \cong \mathbb{R}$, by Theorem 2, we have an isomorphism $H^1_{\mathrm{DR}}(M;\mathbb{R})/\iota[M,U(1)] \xrightarrow{\cong} \mathit{Flat}(M\times U(1))/\mathcal{G}(M\times U(1))$.

Let $[M, U(1)]^0$ be the group of homotopy classes of continuous maps from M to $U(1)$; we denote by $[\varphi]^0$ the homotopy class of a continuous map $\varphi : M \to U(1)$ and by $[f]$ the smooth homotopy class of a smooth map $f : M \to U(1)$. Let $p : [M, U(1)] \to [M, U(1)]^0$ be the homomorphism defined by $p[f] = [f]^0$. Since two smoothly homotopic maps are homotopic, p is well defined. By Bröcker and Jänich (1982), given any continuous map $\varphi : M \to U(1)$ there is a smooth map $f : M \to U(1)$ which is homotopic to φ; this implies that p is surjective. Now let $[f] \in [M, U(1)]$ be an element such that $p[f] = [f]^0 = [1]^0$. Then there is a continuous homotopy $H : M \times I \to U(1)$ such that $H(x, 0) = f(x)$ and $H(x, 1) = 1 \in U(1)$, for all $x \in M$. Consider the closed subset $A = M \times \{0\} \cup M \times \{1\} \subset M \times I$, and $H \mid A$; clearly there is a neighborhood U of A in $M \times I$, and a smooth map $\psi : U \to S^1$ such that $\psi \mid A = H \mid A$. Then by Bröcker and Jänich, there is a smooth map $\tilde{H} : M \times I \to U(1)$ such that $\tilde{H} \mid A = H \mid A$; therefore f and 1 are smoothly homotopic, i.e. $[f] = [1]$, and p is injective. Hence p is an isomorphism. By Spanier (1989), $U(1)$ is an Eilenberg–MacLane space of type $(\mathbb{Z}, 1)$, and the homomorphism $[M, U(1)]^0 \xrightarrow{\sigma} H^1(M; \mathbb{Z})$ given by $\sigma[\varphi]^0 = \varphi^*(c)$ is an isomorphism, where $\varphi^* : H^1(U(1); \mathbb{Z}) \to H^1(M; \mathbb{Z})$ and c is the canonical generator. Therefore, $H^1_{\mathrm{DR}}(M; \mathbb{R})/\iota[M, U(1)] \cong H^1_{\mathrm{DR}}(M; \mathbb{R})/l H^1(M; \mathbb{Z})$, where $l = \iota \circ p^{-1} \circ \sigma^{-1}$. QED

Now we will use these results to study the case $M = \mathbb{R}^2 - \{0\}$.

Proposition 3. *Let $F : \mathbb{R}^2 - \{0\} \to U(1)$ be the smooth map given by $F(x, y) = (x, y)/\|(x, y)\|$, then $F^\#(\mathcal{M}) = i(\frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy)$.*

Proof: A straightforward calculation shows that

$$DF_{(x,y)} = \frac{1}{(x^2 + y^2)^{3/2}} \begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix}.$$

By Definition 1, $\mathcal{M}_g(v) = (dL_{g^{-1}})_g(v)$. Since $U(1) \subset \mathbb{C}$, we have that $\mathcal{M}_g(v) = g^{-1}v$, where the right-hand side is the product of complex numbers. Therefore,

$$\begin{aligned} F^\#(\mathcal{M})(x, y)(t_1, t_2) &= \mathcal{M}_{F(x,y)}\big(DF_{(x,y)}(t_1, t_2)\big) \\ &= \frac{(x, -y)}{(x^2 + y^2)^{1/2}} \frac{1}{(x^2 + y^2)^{3/2}} (y^2 t_1 - xyt_2, -xyt_1 + t_2 x) \\ &= \frac{1}{(x^2 + y^2)^2} (x, -y)(y^2 t_1 - xyt_2, -xyt_1 + t_2 x^2) \\ &= \left(0, \frac{-yt_1 + xt_2}{x^2 + y^2}\right). \end{aligned}$$

QED

Theorem 3. *Let* $\pi : (\mathbb{R}^2 - \{0\}) \times U(1) \to \mathbb{R}^2 - \{0\}$ *be the product bundle. Then there is an isomorphism of groups* $Flat((\mathbb{R}^2 - \{0\}) \times U(1))/\mathcal{G}((\mathbb{R}^2 - \{0\}) \times U(1)) \overset{\cong}{\to} S^1$ *given by* $\langle \omega^\gamma \rangle \mapsto e^{2\pi i \lambda}$, *where* $[\gamma] = \lambda F^*(\mathcal{M})$, $\lambda \in \mathbb{R}$.

Proof: By Theorem 2, we have an isomorphism

$$H^1_{\mathrm{DR}}(\mathbb{R}^2 - \{0\}; \mathbb{R})/\iota[\mathbb{R}^2 - \{0\}, U(1)] \to Flat((\mathbb{R}^2 - \{0\}) \times U(1))/\mathcal{G}((\mathbb{R}^2 - \{0\}) \times U(1))$$

given by $[\overline{\gamma}] \mapsto \langle \omega^\gamma \rangle$. We shall give an isomorphism between the left-hand side and S^1.

In the proof of Corollary 3, we showed that $[\mathbb{R}^2 - \{0\}, U(1)] \cong [\mathbb{R}^2 - \{0\}, U(1)]^0$ and by Spanier (1989), $[U(1), U(1)]^0 \cong \mathbb{Z}$, with canonical generator $[id]$. Now, $U(1)$ is a deformation retract of $\mathbb{R}^2 - \{0\}$ and the map $F : \mathbb{R}^2 - \{0\} \to U(1)$ of Proposition 3 is the retraction, i.e., if $\delta : U(1) \to \mathbb{R}^2 - \{0\}$ is the inclusion, the $F \circ \delta = Id_{U(1)}$ and $\delta^0 F \simeq Id_{\mathbb{R}^2 - \{0\}}$, and so F is in particular a homotopy equivalence. Therefore the homomorphism $F^+ : [U(1), U(1)]^0 \to [\mathbb{R}^2 - \{0\}, U(1)]^0$ given by $F^+[\varphi] = [\varphi \circ F]$ is an isomorphism; hence, $F^+[Id]^0 = [F]^0$ is a generator for $[\mathbb{R}^2 - \{0\}, U(1)]$ and $[F]$ is a generator for $[\mathbb{R}^2 - \{0\}, U(1)]$.

Consider $\iota : [\mathbb{R}^2 - \{0\}, U(1)] \to H^1_{\mathrm{DR}}(\mathbb{R}^2 - \{0\}; \mathbb{R})$ and take $\iota[F] := F^*[\mathcal{M}] = [F^\#(\mathcal{M})]$. By Proposition 3, $F^\#(M) = \frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy$ (we dropped the i, since here we are identifying $i\mathbb{R} \cong \mathbb{R}$). Assume that there is a smooth map $\varphi : \mathbb{R}^2 - \{0\} \to \mathbb{R}$ such that $d \circ \varphi = F^\#(\mathcal{M})$. Let $c(t) = (\cos t, \sin t)$, $0 \le t \le \pi$. Then, by the fundamental theorem of calculus, we would have $\int_c d_0\varphi = 0$, but $\int_c F^\#(\mathcal{M}) = \pi$. Therefore there is no such map φ and hence $[F^\#(\mathcal{M})] = F^*[\mathcal{M}] = \iota[F]$ as an element in $H^1_{\mathrm{DR}}(\mathbb{R}^2 - \{0\}; \mathbb{R})$ is not zero. By De Rham's theorem, $H^1_{\mathrm{DR}}(\mathbb{R}^2 - \{0\}; \mathbb{R}) \cong H^1(\mathbb{R}^2 - \{0\}; \mathbb{R})$, and since $\mathbb{R}^2 - \{0\} \simeq U(1)$, this group is isomorphic to $H^1(U(1); \mathbb{R})$ which is isomorphic to $\mathbb{R}$ (Spanier, 1989). Therefore $F^*[\mathcal{M}]$ is a generator of $H^1_{\mathrm{DR}}(\mathbb{R}^2 - \{0\}; \mathbb{R})$. Thus, we have

$$[\mathbb{R}^2 - \{0\}, U(1)] \cong \mathbb{Z} \xrightarrow{\iota} H^1_{\mathrm{DR}}(\mathbb{R}^2 - \{0\}; \mathbb{R}) \cong \mathbb{R},$$

and any element $[\gamma]$ in $H^1_{\mathrm{DR}}(\mathbb{R}^2 - \{0\}; \mathbb{R})$ can be written as $[\gamma] = \lambda F^\#(\mathcal{M})$. Define a homomorphism $H^1_{\mathrm{DR}}(\mathbb{R}^2 - \{0\}; \mathbb{R}) \to S^1$ by $[\gamma] = \lambda F^\#(\mathcal{M}) \mapsto e^{2\pi i \lambda}$. The homomorphism is clearly surjective, and its kernel is the image of ι because $e^{2\pi i \lambda} = 1 \Leftrightarrow \lambda \in \mathbb{Z} \Leftrightarrow \lambda F^\#(\mathcal{M})$ belongs to the image of ι. Therefore we have an isomorphism $H^1_{\mathrm{DR}}(\mathbb{R}^2 - \{0\}; \mathbb{R})/\iota[\mathbb{R}^2 - \{0\}, U(1)] \xrightarrow{\cong} S^1$ given by $[\overline{\gamma}] \mapsto e^{2\pi i \lambda}$, where $[\gamma] = \lambda F^\#(\mathcal{M})$. The composite of this isomorphism with the one defined above, gives the isomorphism of the theorem. QED

5. HOLONOMY GROUPS

In this section we will calculate the holonomy group of each gauge equivalence class of flat connections on $M \times U(1) \xrightarrow{\pi} M$, where $M = \mathbb{R}^2 - \{0\}$. Notice that, by the lemma below, all connections in the same gauge equivalence class have isomorphic holonomy groups.

Definition 5. Let $p : P \to M$ be a principal G-bundle, ω a connection on P, and a a point in P. We denote by $\Phi_\omega(a) \subset G$ the holonomy group of ω based at the point a.

Lemma 8. *Let $p : P \to M$ be a principal G-bundle and ω_1, ω_2 two connections on P such that they are gauge equivalent. Then, for each a in P, there exists a g in G such that $\Phi_{\omega_1}(a) = g^{-1}\Phi_{\omega_2}(a)g$; in particular both holonomy groups are isomorphic.*

Proof: Let φ be a gauge transformation such that $\varphi^*(\omega_1) = \omega_2$. Then it is easy to see that $(d\varphi)_a(H^1_a) = H^2_{\varphi(a)}$, where H^i is the horizontal subspace defined by the connection ω_i. Let σ be a piecewise smooth loop at a point x_0 in M and let a_0 be in $p^{-1}(x_0)$. Then, using ω_1, we define an equivariant diffeomorphism $\overline{\sigma}_1 : p^{-1}(x_0) \to p^{-1}(x_0)$, and hence, an element $g_{1,\sigma}$ in the gauge group $\Phi_{\omega_1}(a_0)$, given by $\overline{\sigma}_1(a_0) = a_0 \cdot g_{1,\sigma}$. For ω_2 we take $\varphi(a_0)$ in $p^{-1}(x_0)$, and using ω_2, we have an element $g_{2,\sigma}$ such that $\overline{\sigma}_2(\varphi(a_0)) = \varphi(a_0) \cdot g_{2,\sigma}$. Now, $\overline{\sigma}_1(a_0)$ is obtained by taking the unique ω_1-horizontal lifting $\tilde{\sigma}_1$ of σ such that $\tilde{\sigma}_1(0) = a_0$, and setting $\overline{\sigma}_1(a_0) = \tilde{\sigma}_1(1)$. Notice that since $p \circ \varphi \circ \tilde{\sigma}_1 = p \circ \tilde{\sigma}_1 = \sigma$, then $\varphi \circ \tilde{\sigma}_1$ is a lifting of σ such that $\varphi \circ \tilde{\sigma}_1(0) = \varphi(a_0)$, and hence $\overline{\sigma}_2(\varphi(a_0)) = \varphi \circ \tilde{\sigma}_1(1)$, provided $\varphi \circ \tilde{\sigma}_1$ is ω_2-horizontal. But $\dot{\tilde{\sigma}}_1(t) \in H^1_{\tilde{\sigma}_1(t)}$, and therefore $(\varphi \circ \dot{\tilde{\sigma}}_1)(t) = (d\varphi)_{\tilde{\sigma}_1(t)}(\tilde{\sigma}_1(t)) \in H^2_{\varphi\circ\tilde{\sigma}_1(t)}$. Since $\overline{\sigma}_2(\varphi(a_0)) = \varphi(a_0) \cdot g_{2,\sigma}$ and $\overline{\sigma}_2(\varphi(a_0)) = \varphi \circ \tilde{\sigma}(1) = \varphi(a_0 \cdot g_{1,\sigma}) = \varphi(a_0) \cdot g_{1,\sigma}$, and the action is free, we have that $g_{2,\sigma} = g_{1,\sigma}$, i.e. $\Phi_{\omega_1}(a_0) \subset \Phi_{\omega_2}(\varphi(a_0))$. Using φ^{-1}, the same proof shows that $\Phi_{\omega_2}(\varphi(a_0)) \subset \Phi_{\omega_1}(a_0)$. Hence $\Phi_{\omega_1}(a_0) = \Phi_{\omega_2}(\varphi(a_0))$. Since a_0 and $\varphi(a_0)$ are in $p^{-1}(x_0)$, and the action is transitive on fibers, there is a g in G such that $a_0 \cdot g = \varphi(a_0)$. By Kobayashi and Nomizu (1963), we have that $\Phi_{\omega_2}(\varphi(a_0)) = g^{-1}\Phi_{\omega_2}(a_0)g$; therefore, $\Phi_{\omega_1}(a_0) = g^{-1}\Phi_{\omega_2}(a_0)g$. QED

Theorem 4. *Let $\pi : (\mathbb{R}^2 - \{0\}) \times U(1) \to \mathbb{R}^2 - \{0\}$ be the product bundle. Let $\langle\omega^\gamma\rangle$ be the gauge equivalence class of any flat connection ω^γ on π. Then $\Phi_{\langle\omega^\gamma\rangle}$ is the subgroup of S^1 generated by $e^{2\pi i\lambda}$, where $[\gamma] = \lambda F^\#(\mathcal{M})$, $\lambda \in \mathbb{R}$, i.e., $\Phi_{\langle\omega^\gamma\rangle} = \{e^{2\pi in\lambda} \mid n \in \mathbb{Z}\}$.*

Proof: By Theorem 3, we have an isomorphism

$$Flat((\mathbb{R}^2 - \{0\}) \times U(1))/\mathcal{G}((\mathbb{R}^2 - \{0\}) \times U(1)) \xrightarrow{\cong} S^1$$

given by $\langle\omega^\gamma\rangle \mapsto e^{2\pi i\lambda}$, where $[\gamma] = \lambda F^*(\mathcal{M})$ is in $H^1_{\mathrm{DR}}(\mathbb{R}^2 - \{0\}; \mathbb{R})$. We are going to calculate the holonomy of the connection ω^γ, where $\gamma = \lambda F^\#(\mathcal{M})$. Since $[\gamma] = [\lambda F^\#(\mathcal{M})] = \lambda[F^\#(\mathcal{M})] = \lambda F^*(\mathcal{M})$, then $\langle\omega^\gamma\rangle \mapsto e^{2\pi i\lambda}$. We take as base point in $\mathbb{R}^2 - \{0\}$, the point $(1, 0)$; and as base point in the total space, the point $((1, 0), e)$, where e is the neutral element in $U(1)$. Since ω^γ is a flat connection, by Kobayashi and Nomizu (1963), parallel displacement defines a surjective homomorphism from $\Pi_1(\mathbb{R}^2 - \{0\}, (1, 0))$ to $\Phi_{\omega^\gamma}((1, 0), e)$. By Spanier (1989), $\Pi_1(\mathbb{R}^2 - \{0\}, (1, 0)) \cong \mathbb{Z}$, and we can take as a generator the curve $\sigma(t) = e^{2\pi i t}$, where $t \in [0, 1]$. In order to find the ω^γ-horizontal lifting of σ, we need to determine the horizontal subspaces of ω^γ. By Lemma 2, we have that $\omega_{((x,y),g)}(u, v) = \gamma_{(x,y)}(u) + g^{-1}v$, where $(x, y) \in \mathbb{R}^2 - \{0\}$ and $g \in U(1)$. Hence, $H_{((x,y),g)} = \{(u, v) \in T_{(x,y)}(\mathbb{R}^2 - \{0\}) \times T_g U(1) \mid g^{-1}v = -\gamma_{(x,y)}(u)\}$. Since $\gamma = \lambda F^\#(\mathcal{M})$, we have, by Proposition 3, that

$$H_{((x,y),g)} = \left\{((t_1, t_2), v) \mid g^{-1}v = i\lambda\left(\frac{yt_1}{x^2 + y^2} - \frac{xt_2}{x^2 + y^2}\right)\right\},$$

where $(t_1, t_2) \in \mathbb{R}^2 = T_{(x,y)}(\mathbb{R}^2 - \{0\})$, $g \in U(1)$, and v is orthogonal to g. We define $\tilde{\sigma} : [0, 1] \to (\mathbb{R}^2 - \{0\}) \times U(1)$, by $\tilde{\sigma}(t) = (\sigma(t), \alpha(t))$, where $\alpha(t) = e^{-2\pi i\lambda t}$. Clearly $\tilde{\sigma}$ is a lifting of σ and

$$\begin{aligned}\dot{\tilde{\sigma}}(t) &= (2\pi i\, e^{2\pi i t}, -2\pi i\lambda\, e^{-2\pi i\lambda})\\ &= ((-2\pi \sin 2\pi t, 2\pi \cos 2\pi t), -2\pi i\lambda\, e^{-2\pi i\lambda t}).\end{aligned}$$

By the calculation above, the ω^γ-horizontal subspace at a point $(x, y) = e^{2\pi i t} = (\cos 2\pi t, \sin 2\pi t)$, $g = e^{-2\pi i\lambda}$ is given by

$$\{((t_1, t_2), v) \mid (\cos 2\pi\lambda t, \sin 2\pi\lambda t)v = i\lambda(t_1 \sin 2\pi t - t_2 \cos 2\pi t)\}$$

and an easy calculation shows that $\dot{\tilde{\sigma}}(t)$ is in the ω^γ-horizontal subspace. Therefore, since $\tilde{\sigma}(0) = ((1, 0), e)$ and $\tilde{\sigma}(1) = ((1, 0), e^{-2\pi i\lambda})$, one has that the element of $\Phi_{\langle\omega^\gamma\rangle} \subset S^1$, associated to the loop σ is $e^{-2\pi i\lambda}$. Since $\Pi_1(\mathbb{R}^2 - \{0\}, (1, 0)) \cong \mathbb{Z}$, $\Phi_{\langle\omega^\gamma\rangle}$ is the subgroup generated by this element, which is the same as the subgroup generated by $e^{2\pi i\lambda}$. QED

Corollary 4. *Let ω^γ be any flat connection on $\pi : (\mathbb{R}^2 - \{0\}) \times U(1) \to \mathbb{R}^2 - \{0\}$, where $[\gamma] = \lambda F^*(\mathcal{M})$. If λ is rational, then its holonomy group is a finite cyclic group. If λ is irrational, then its holonomy group is isomorphic to $\mathbb{Z}$ and it is dense in S^1.*

Proof: By Theorem 4, the holonomy group of ω^γ is the subgroup of S^1 generated by $e^{2\pi i\lambda}$. If λ is rational, $\lambda = p/q$, then $q\lambda$ is an integer and $e^{2\pi i q\lambda} = 1$. Hence $\Phi_{\langle\omega^\gamma\rangle}$ is finite cyclic; in particular if λ is an integer, the holonomy group is trivial. Assume now that λ is irrational; if there is a nonzero integer n such that

$e^{2\pi i n\lambda} = 1$, then $n\lambda = m \in \mathbb{Z}$, i.e. λ is rational, which is a contradiction, and so no such integer exists and hence $\Phi_{\langle \omega^\gamma \rangle} \cong \mathbb{Z}$. By Auslander (1988), this group is dense in S^1.
QED

6. THE A-B CONNECTION

Although the presence of the geometrical connection discussed in the previous section is fundamental and completely natural, it basically depends only on the topology of the base space, because by Theorem 2 even if we take as structural group $G = \mathbb{R}$ we still have flat connections which are not gauge equivalent to the trivial connection ($H^1_{DR}(\mathbb{R}^{2*}, \mathbb{R})/[\mathbb{R}^{2*}, \mathbb{R}] \cong \mathbb{R}/0 \cong \mathbb{R}$); from the physical point of view this is not enough. According to the discussion in Section 3, based on the Feynman's path integral approach to quantum mechanics, the crucial factor $\frac{\Phi}{2\pi}$, where Φ is the magnetic flux inside the solenoid, has to be considered, leading to the *Aharonov–Bohm connection*

$$A = \frac{\Phi}{2\pi} \frac{x\,dy - y\,dx}{x^2 + y^2}. \tag{6.1}$$

Summing up, locally we have that

$$A = \frac{\Phi}{2\pi} d\varphi, \tag{6.2}$$

where $\varphi \in (0, 2\pi)$ is the local polar coordinate.

Defining

$$A_0 = \frac{\hbar c}{|e|} d\varphi, \tag{6.3}$$

we have

(i) $[A_0] = \{A_0 + d\alpha\}_{\alpha \in C^\infty(\mathbb{R}^{2*}, \mathbb{R})}$ is a generator of the cohomology of $\mathbb{R}^{2*}$ in dimension 1, which is isomorphic to $\mathbb{R}$: $H^1(\mathbb{R}^{2*}; \mathbb{R}) \cong \{\lambda[A_0] = [\lambda A_0]\}_{\lambda \in \mathbb{R}}$. Notice that any function in $C^\infty(\mathbb{R}^{2*}, \mathbb{R})$ gives rise to an element of $C^\infty(\mathbb{R}^{2*}, S^1)$, *the gauge group of the bundle*, through $\alpha \mapsto e^{i\alpha}$; however $C^\infty(\mathbb{R}^{2*}, \mathbb{R})$ does not exhaust $C^\infty(\mathbb{R}^{2*}, S^1)$ since any differentiable map $\gamma : \mathbb{R}^{2*} \to S^1$ homotopic to $e^{in\varphi}$ with $n \neq 0$ cannot be lifted to a map $\alpha : \mathbb{R}^{2*} \to \mathbb{R}$ such that $\gamma = e^{i\alpha}$.

(ii) *The moduli space of flat connections in the product bundle ξ is the circle*: if n is an integer, then $(n+1)A_0 = A_0 + nA_0 = A_0 + n\frac{\hbar c}{|e|} d\varphi = A_0 + d(n\frac{\hbar c}{|e|}\varphi)$ is flat and is a gauge transform of A_0; then

$$\begin{aligned} &\left\{\begin{array}{c}\text{gauge equivalence classes}\\ \text{of flat connections on } \xi\end{array}\right\} \leftrightarrow \left\{\lambda \frac{x\,dy - y\,dx}{x^2 + y^2}\right\}_{\lambda \in (0,1)} \\ &\qquad \leftrightarrow \{e^{2\pi i \lambda}\}_{\lambda \in (0,1)} \leftrightarrow \mathbb{R}/\mathbb{Z} \cong S^1. \end{aligned} \tag{6.4}$$

In physical terms the relevant connection is A_0, and one has the isomorphism

$$\begin{array}{r}\{\text{gauge equivalence classes}\\ \text{of flat connections on } \xi\}\end{array} \leftrightarrow \{[\lambda A_0]\}_{\lambda\in(0,1)} \leftrightarrow \frac{\mathbb{R}}{\mathbb{Z}} \cong S^1, \tag{6.5}$$

where

$$[\lambda A_0] = \{\lambda A_0 + f^{-1} df\}_{f\in C^\infty(R^{2*},S^1)}. \tag{6.6}$$

A_0 involves quantum mechanics ($\hbar$), special relativity (c), electromagnetism ($|e|$), and differential geometry ($d\varphi$); therefore, it can be considered a "natural" object associated with the plane minus a point, generating the nontrivial part of its cohomology, and *all* gauge nonequivalent vacuum potentials.

Finally, it is important to remark that we have two holonomies: a geometrical holonomy and a physical holonomy, and they are related by the formula

$$\text{physical holonomy}(A) = \text{geometrical holonomy}(|e|A), \tag{6.7}$$

where $|e|$ is the absolute value of the electromagnetic coupling.

7. FINAL REMARK

When the magnetic flux is quantized in units of Φ_0, and therefore the A-B effect vanishes, A of (6.1) coincides with the asymptotic value at long distances of the *vortex solution* to the abelian Higgs model: charged scalar electrodynamics with spontaneous symmetry breaking, in which both the time independent scalar and electromagnetic fields are defined in three-dimensional space but with cylindrical symmetry in one direction (de Vega and Schaposnik, 1976; Nielsen and Olesen, 1973). In this case, the field configurations are smooth, and no extension of Stokes' theorem is required, the quantization of the magnetic flux being obtained by integrating on the circle at infinity the boundary condition on the magnetic potential.

ACKNOWLEDGMENT

M. Socolovsky thanks the Instituto de Astronomía y Física del Espacio (Universidad de Buenos Aires-CONICET), Argentina, where part of this work was done, for its hospitality.

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